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Variation of Geometric Invariant Theory Quotients

Igor V. Dolgachev

University of Michigan, Ann Arbor, MI 48109 , USA

1. This is a report on a joint work with Yi Hu (University of Michigan). Let G be a reductive algebraic group acting on a normal irreducible quasi-projective algebraic variety X, both defined over an algebraically closed field k. Given a G-linearized line bundle L on X, it defines an open subset of stable points $X^s(L) \subset X$ such that the orbit space $X^s(L)/G$ exists in the category of quasi-projective varieties. A larger set $X^{ss}(L)$ of semi-stable points allows one to define a quasi-projective variety $X^{ss}(L)//G$ which contains $X^s(L)/G$ as an open subset and parametrizes closed orbits of G in $X^{ss}(L)$. It is a projective variety if X is projective. In this note we want to approach the following fundamental question. What happens to the quotients $X^s(L)/G$ and $X^{ss}(L)//G$ when we vary L? Intuitively, it is clear that all the quotients are birationally isomorphic unless they have different dimensions. We want to describe precisely the corresponding birational transformations. This problem is analogous to the problem of the variation of symplectic reductions of a symplectic manifold M with respect to an action of a compact Lie group K. Recall that if $K \times M \to M$ is a Hamiltonian action with the moment map $\Phi : M \to Lie(K)^*$, then for any point $p \in \Phi(M)$, the orbit space $\Phi^{-1}(K \cdot p)/K$ is the symplectic reduction of M by K with respect to the point p. If K is a torus, $M = X$ with the symplectic form defined by the Chern form of L, and K acts on X via the restriction of an algebraic action of its complexification T, then the choice of a rational point $p \in \Phi(X)$ corresponds to the choice of a T-linearization on L, and the symplectic reduction $Y_p = \Phi^{-1}(p)/K$ is isomorphic to the GIT quotient $X^{ss}(L)//T$. It turns out in this case that if we let p vary in a connected component F of the set of regular values of the moment map, the symplectic reductions Y_p are all diffeomorphic to the same manifold Y_F. However if we let p cross a wall separating one connected component from another, the reduction Y_p undergoes a very special surgery which is similar to a birational transformation known as a flip. This was shown in a work of V. Guillemin and S. Sternberg [**GS**]. In a purely algebraic setting this was proven independently (and about the same time) by M. Brion and C. Procesi [**BP**] (cf. also [**Hu**]). It turns out that one can define a canonical morphism of algebraic varieties $\pi_{G,G'} : Y_G \to Y_{G'}$, where G and G' are open faces of some F as above with G' contained in the closure of G. If G' is not contained in the boundary of $\Phi(X)$, then $\pi_{G,G'}$ is a birational morphism whose fibers are fibration towers of weighted projective spaces.

2. . Let L be a G–linearized ample line bundle on a quasi-projective normal algebraic variety X. We say that L is *G–effective* if there exists $s \in \Gamma(X, L^n)^G$ for some $n > 0$ such that $X_s = \{x \in X : s(x) \neq 0\}$ is non-empty and affine. It follows from the definition of semi-stable points that a G-linearized bundle L is G-effective if and only if $X^{ss}(L) \neq \emptyset$. Let $Pic^G(X)$ denote the group of isomorphism classes of G-linearized line bundle. The image of the canonical forgetting homomorphism $Pic^G(X) \to Pic(X)$ is a subgroup of finite index and its kernel is a finitely generated abelian group isomorphic to the group of characters $\mathrm{X}(G) = Hom_k(G, G_m)$ of G if X is projective(cf.[**KKV**]). Let $Pic^G(X)_+$ be the subset of $Pic^G(X)$ which consists of isomorphism classes of G–effective linearized line bundles. Obviously it is a semigroup in $\mathrm{Pic}^G(X)$.

Definition 1. The *moment cone* $C^G(X)$ of (X, G) is the closed convex cone in $Pic^G(X)_{\mathbf{R}}$ spanned by the image of $\mathrm{Pic}^G(X)_+$ in $\mathrm{Pic}^G(X)$.

In most applications $Pic(X)$ is finitely generated so that $C^G(X)$ is a closed convex cone in a finite-dimensional vector space. If X is projective we can replace Pic$X$ by the Neron-Severi group NS(X) by using the fact that $X^{ss}(L) = X^{ss}(L')$ if L and L' are ample G–linearized line bundles which are algebraically equivalent ([**MF**], p. 48).

Example 1. Let G be a complex n-torus which acts linearly on a vector space V and projectively linearly on the projectivization $X = P(V)$. Then the compact real form T_c of T acts symplectically on X with respect to the Fubini-Studi symplectic structure on X. The image of the associated moment map $\Phi : X \to \mathbf{R}^n$ is a rational polyhedron equal to the convex hull of weights of T in V. The representation ρ_0 of T on V is defined by a choice of linearization σ_0 on $L = \mathcal{O}_{\mathbf{P}(V)}(1)$. We have $X^{ss}(L, \sigma_0)$ contains $G\Phi^{-1}(0)$ as the set of closed orbits ([**Ne**]). If (L^k, σ) is an arbitrary linearization on L^k, then the corresponding representation of T is isomorphic to the representation $\rho_0^{\otimes k} \otimes \chi$ for some character χ of T. This allows one to identify $Pic^G(X)_+$ with the set $\{(k, \chi) \in \mathbf{Z}_{\geq 0} \times Char(T) : 0 \in \chi + k\Phi(X)\}$. This identifies $C^G(X)$ with $\{(\lambda, a) \in \mathbf{R}_{\geq 0} \times \mathbf{R}^n : -\lambda a \in \Phi(X)\}$ which is a cone over $-\Phi(X)$.

Assume now that X is projective. Let λ be a 1-parameter subgroup of G, x be a point of X and $x_0 = \lim_{\lambda \to 0} \lambda x$. Then x_0 is a fixed point of λ. For any $L \in Pic^G(X)$, λ acts on the fiber L_{x_0} via a character which can be identified with some integer $\mu^L(x, \lambda)$. We set $M^L(x) = sup_\lambda\ \mu^L(x, \lambda)/\|\lambda\|$, where $\|\|$ is a positive-definite K-invariant norm on the set of one–parameter subgroups of G, K being a compact form of G (cf. [**Ne**]). If L is an ample G–linearized line bundle, the Mumford-Hilbert numerical criterion of stability asserts that $x \in X^{ss}(L)$ (resp. $x \in X^s(L)$) if and only if $M^L(x) \leq 0$ (resp. $M^L(x) < 0$). We extend the function $L \to M^L(x)$ to $\Pi^G(X) \otimes \mathbf{Q}$ by setting $M^{\alpha L}(x) = \alpha M^L(x)$ for any rational number α. Then we extend this function by continuity to $Pic^G(X) \otimes \mathbf{R}$.

Definition 2. The zero set $H(x)$ of the function $M^\bullet(x)$ in $C^G(X)$ for some point x with stabilizer of positive dimension is called a *wall*. A connected non-empty component of $C^G(X) \setminus (\cup_{x\in X} H(x))$ is called a *chamber*. We say that $H(x)$ is an *interior wall* if it separates two chambers. A *face* F is the interior of intersection of a the closures of a finite set of chambers .

It is easy to see that in the situation of Example 1 the closure of each face F is a rational polyhedral cone. This leads to a number of fundamental questions regarding the cone $C^G(X)$ and its faces F (e.g., finiteness of chambers, their structure and etc.).

For any rational point $l \in C^G(X)$ we can define the set $X^{ss}(l)$ (resp. $X^s(l)$) as being equal to $X^{ss}(L)$ (resp. $X^s(L)$), where the class of L in $C^G(X)$ is rationally proportional to l.

Proposition 1. *Let F be an iterior face. Two rational points l and l' belong to F if and only if $X^{ss}(l) = X^{ss}(l')$. If F is a chamber, then two rational points l and l' belong to F if and only if $X^s(l) = X^s(l')$.*

Proof. Easily follows from the Hilbert-Mumford numerical criterion of stability.

Using Proposition 1, we can define chambers and faces in $C^G(X)$, where X is not necessary projective. A face is an equivalence class in $NS^G(X)$ with respect to the equivalence relation $L \sim L'$ if and only if $X^{ss}(L) = X^{ss}(L')$. A chamber is a face that contains some L with $X^{ss}(L) = X^s(L)$.

Proposition 2. *Let U be an open G-invariant subset of X, and let $C^G(X, U)$ be the set of all L from $C^G(X)$ such that $X^{ss}(L)$ is contained in U. Then the natural map $C^G(X, U) \to C^G(U)$ is surjective. The pre-image of any face in $C^G(U)$ is contained in a face of $C^G(X)$.*

We let X_F be the set $X^s(l)$, where l is a rational point in F. Let $\pi_F : X_F \to Y_F = X_F/G$ be the corresponding geometric quotient morphism. For any variety Y we denote by $Pic(Y)^+$ (resp.$Pic(Y)^{++}$) the semi-group of effective (resp. ample) line bundles.

Proposition 3. *Let F be a chamber. Assume the following conditions are satisfied:*

(i) X is nonsingular;
(ii) the complement $X - X_F$ is of codimension ≥ 2. Then the map $L \to (\pi_F)^G(L|X_F)$ defines an injective map $C^G(X) \to Pic(Y_F)^+_{\mathbf{R}}$. The image of of F is contained in $Pic(Y_F)^{++}_{\mathbf{R}}$.
Proof. Since G acts on X_F with finite isotropy subgroups, the canonical map Pic$(Y_F) \to Pic^G(X_F)$, $M \to \pi_F^*(M)$ is injective and its image is a subgroup of finite index **[KKV]**. This establishes a bijective map from Pic$^G(X_F)_{\mathbf{R}}$ to $Pic(Y_F)_{\mathbf{R}}$. By conditions (i) and(ii), the restriction map $Pic^G(X) \to Pic^G(X_F)$ is bijective. This induces a bijection between $Pic^G(X)_{\mathbf{R}}$ and $Pic(Y_F)_{\mathbf{R}}$. By the projection formula, $M \cong (\pi_F)^G_*(\pi_F^*(M))$. Since $\Gamma(X, L)^G \neq \{0\}$ if and only if $\Gamma(Y_F, (\pi_F)^G_*(L|X_F)) \neq \{0\}$, we get an injection $C^G(X) \to Pic(Y_F)^+_{\mathbf{R}}$. If $L \in F$, then its restriction to X_F is ample. Thus some multiple of L descends to an ample line bundle on Y_F. Now all the assertions are proved.

Example 2. According to D. Cox [Co], any toric variety Y_Σ defined by a simplicial fan Σ is isomorphic to a geometric quotient of an open subset U of $X = \mathbf{C}^n$ with complement of codimension ≥ 2 by a complex torus T with the character group Char(T) equal to Pic(Y_Σ). By [**MF**], p. 41, if Y_Σ is projective, $U = X^{ss}(L) = X^s(L)$ for some T-linearized line bundle L. Therefore $U = X_F$ for some chamber F in $C^T(X)$ and we can apply Proposition 3. In fact one can show more, namely that the map $C^T(X) \to Pic(Y_\Sigma)^+_{\mathbf{R}}$ is bijective. Here we use the fact that the complement of a hypersurface in X is affine. Now each chamber F' in $C^T(X)$ defines a geometric quotient X'_F/T which is a toric variety $Y_{\Sigma'}$ corresponding to a fan Σ' with the same 1-skeleton as Σ. The closure $\overline{F'}$ of the cone F' is ! identified with the cone of nef d $\overline{F'}$ and their faces form the Gelfand-Kapranov-Zelevinski decomposition of the cone $C^T(X)$ in the sense of [**OP**].

3. Main Theorems. We are now ready to state our main results.

Theorem 1. (Variation of quotients). *Let G be a reductive algebraic group acting on a nonsingular projective variety X. Let H be an interior face in the cone $C^G(X)$ and l_0 be a rational point in H . Let F^+ and F^- be two chambers such that there is a straight path going from one to another and passing through l_0. Then there are two birational morphisms $f^+ : Y_{F^+} \to X^{ss}(l_0)//G$ and $f^- : Y_{F^-} \to X^{ss}(l_0)//G$ so that if setting Σ_0 to be $(X^{ss}(L_0) - X^s(L_0))//G$, we have*

(i) f^+ and f^- are isomorphisms over the complement to Σ_0;
(ii) over each connected component of Σ_0, the maps $f^\pm$ are towers of weighted $\mathbf{P}^{d_\pm}$-bundles.

Theorem 2. *Assume that any wall is contained in a linear hyperplane. Then the number of interior faces and walls is finite. Each chamber is a convex polyhedral cone in the interior of the cone $C^G(X)$.*

Example 4. Let C be a Riemann surface of genus g, Λ a line bundle on C, and (E, Ψ) be a pair consisting of a rank 2 vector bundle E on C with determinant Λ and a section $\Psi \in \Gamma(E) \setminus \{0\}$. Let σ be a positive rational number. A pair (E, Ψ) is called σ-semistable if for all line bundles $L \subset E$, deg$L \leq 1/2$deg$E - \sigma$ if $\Psi \in \Gamma(L)$, and deg$L \leq 1/2$deg$E + \sigma$ if $\Psi \notin \Gamma(L)$. In [**Th**] the moduli space $\mathbf{M}(\sigma, \Lambda)$ of σ-semistable pairs (E, Ψ) was constructed as some GIT quotient of the product X of two projective spaces by the action of $SL(N)$, $N = h^0(E)$. As was observed first by M. Thaddeus (cf. [**Re**]), different spaces $\mathbf{M}(\sigma, \Lambda)$ correspond to choice of different $L \in C^G(X)$. It is easy to see that $C^G(X) \subset \mathbf{R}^2$ can be identified with the cone over the interval $[0, 1/2(d-1)]$. The chambers F are the cones over the intervals $(\max(0, 1/2d - \alpha - 1), 1/2d - \alpha)$, where $d =$deg(Λ) and α is an integer between 0 and $1/2(d-1)$. The limit quotient is related to the following variety constructed by A. Bertram [**Be**]. One embeds the curve C into the space $\mathbf{P}(H^1(C, \Lambda^{-1}))$ by using the complete linear system $|\Lambda \otimes K_C|$. Then starting with blowing up C, he blows up the proper transform of the 1-secant variety of C, then the proper transform of the 2-secant variety and so on. After $1/2(d-1)$ steps one obtains a smooth variety Y which dominates all the moduli spaces $\mathbf{M}(\sigma, \Lambda)$. By the universality property of projective limits, it is mapped birationally to the limit quotient.

Example 5. One can treat in the same way the moduli spaces of parabolic bundles on a Riemann surface of genus g (cf. [**BH**]). For example, if $g = 0$, we recover the space $U(\alpha)$

from [**Ba**] as GIT quotients in Example 3. The limit quotient in this case coincides with the Mumford-Knudsen moduli space and provides yet another interesting interpretation of the Mumford-Knudsen compactification.

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